Multi-rate Signal Processing
8. General Alias-Free Conditions for Filter Banks
9. Tree Structured Filter Banks and Multiresolution Analysis

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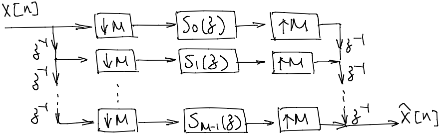
Recall: Simple Filter Bank Systems

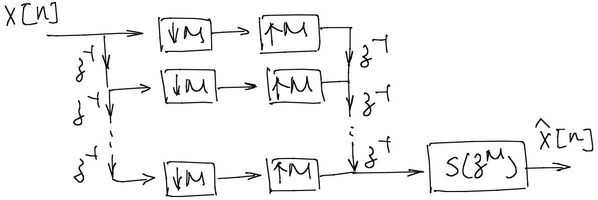


If all $S_k(z)$ are identical as $S(z)$:

$\mathbb{P}(z) = S(z)\mathbb{I}$

$\Rightarrow \hat{X}(z) = z^{-(M-1)} S(z^M) X(z)$

Alias Free



General Alias-free Condition

Recall from Section 7: The condition for alias cancellation in terms of $\mathcal{H}(z)$ and $\underline{\mathbb{f}}(z)$ is

$$\mathcal{H}(z)\underline{\mathbb{f}}(z) = \underline{\mathbb{t}}(z) = \begin{bmatrix} MA_0(z) \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Theorem

A M-channel maximally decimated filter bank is alias-free

iff the matrix $\mathbb{P}(z) = \mathbb{R}(z)\mathbb{E}(z)$ is **pseudo circulant**.

[Readings: PPV Book 5.7]

Circulant and Pseudo Circulant Matrix

(right-)circulant matrix

$$\begin{bmatrix} P_0(z) & P_1(z) & P_2(z) \\ P_2(z) & P_0(z) & P_1(z) \\ P_1(z) & P_2(z) & P_0(z) \end{bmatrix}$$

Each row is the right circular shift of previous row.

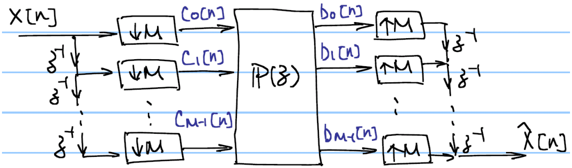
pseudo circulant matrix

$$\begin{bmatrix} P_0(z) & P_1(z) & P_2(z) \\ z^{-1}P_2(z) & P_0(z) & P_1(z) \\ z^{-1}P_1(z) & z^{-1}P_2(z) & P_0(z) \end{bmatrix}$$

Adding z^{-1} to elements below the diagonal line of the circulant matrix.

- Both types of matrices are determined by the 1st row.
- Properties of pseudo circulant matrix (or as an alternative definition): Each column as up-shift version of its right column with z^{-1} to the wrapped entry.

Insights of the Theorem



Denote $\mathbb{P}(z) = [P_{s,\ell}(z)]$.

(Details) For further exploration: See PPV Book 5.7.2 for detailed proof.

Examine the relation between $\hat{X}(z)$ and $X(z)$, and evaluate the gain terms on the aliased versions of $X(z)$.

Overall Transfer Function

The overall transfer function $T(z)$ after aliasing cancellation: $\hat{X}(z) = T(z)X(z)$, where

$$T(z) = z^{-(M-1)}\{P_{0,0}(z^M) + z^{-1}P_{0,1}(z^M) + \cdots + z^{-(M-1)}P_{0,M-1}(z^M)\}$$

(Details) For further exploration: See PPV Book 5.7.2 for derivations.

Most General P.R. Conditions

Necessary and Sufficient P.R. Conditions

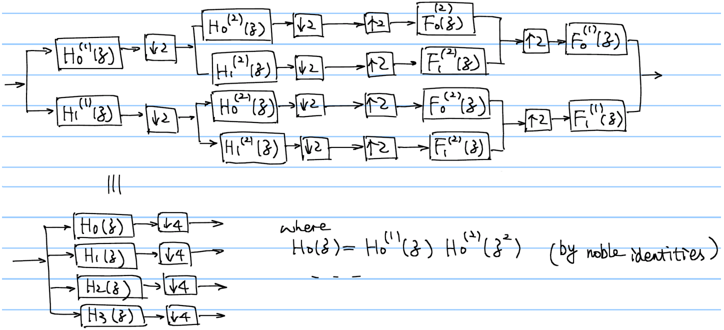
$$\mathbb{P}(z) = cz^{-m_0} \begin{bmatrix} 0 & \mathbb{I}_{M-r} \\ z^{-1}\mathbb{I}_r & 0 \end{bmatrix} \text{ for some } r \in 0, \ldots, M-1.$$

When $r = 0$, $\mathbb{P}(z) = \mathbb{I} \cdot cz^{-m_0}$, as the sufficient condition seen in §I.7.3.

(Details)

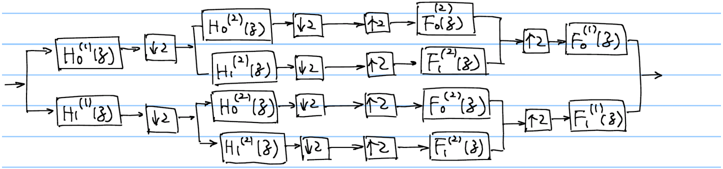
(Binary) Tree-Structured Filter Bank

A multi-stage way to build M-channel filter bank:
Split a signal into 2 subbands $\Rightarrow$ further split one or both subband signals into 2 $\Rightarrow \cdots$



Question: Under what conditions is the overall system free from aliasing? How about P.R.?

(Binary) Tree-Structured Filter Bank

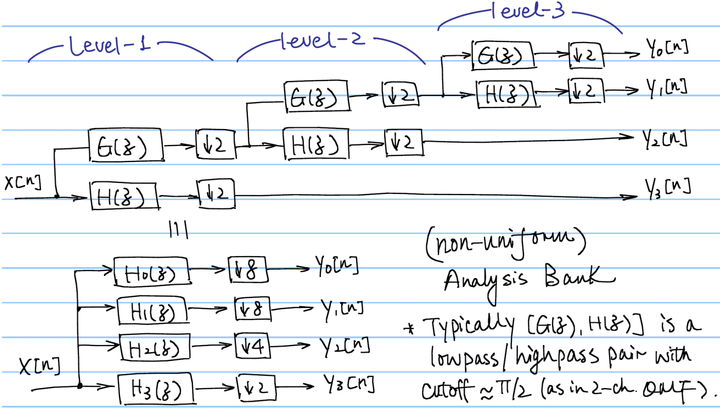


- Can analyze the equivalent filters by noble identities.
- If a 2-channel QMF bank with $H_0^{(K)}(z)$, $H_1^{(K)}(z)$, $F_0^{(K)}(z)$, $F_1^{(K)}(z)$ is alias-free, the complete system above is also alias-free.
- If the 2-channel system has P.R., so does the complete system.

[Readings: PPV Book 5.8]

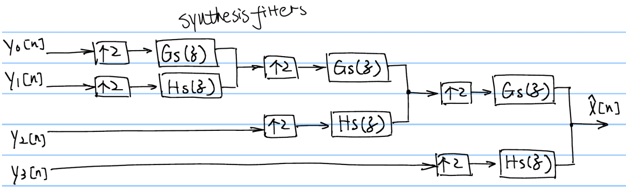
Multi-resolution Analysis: Analysis Bank

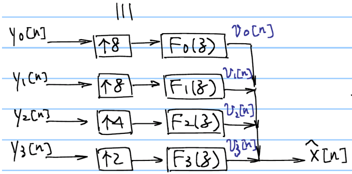
Consider the variation of the tree structured filter bank (i.e., only split one subband signals)



$$H_0(z) = G(z)G(z^2)G(z^4) \Rightarrow H_0(\omega) = G(\omega)G(2\omega)G(2^2\omega)$$

Multi-resolution Analysis: Synthesis Bank





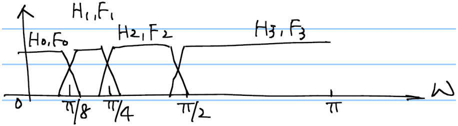
Synthesis Bank

* $\{v_k[n]\}$ have the same sampling rate as $x[n]$ and $\hat{x}[n]$

They are called the multiresolution components.

Discussions

(1) The typical frequency response of the equivalent analysis and synthesis filters are:



(2) The multiresolution components $v_k[n]$ at the output of $F_k(z)$:

- $v_0[n]$ is a lowpass version of $x[n]$ or a "coarse" approximation;
- $v_1[n]$ adds some high frequency details so that $v_0[n] + v_1[n]$ is a finer approximation of $x[n]$;
- $v_3[n]$ adds the finest ultimate details.

Discussions

(3) If 2-ch QMF with $G(z)$, $F(z)$, $G_s(z)$, $F_s(z)$ has P.R. with unit-gain and zero-delay, we have $x[n] = x[n]$.

(4) For compression applications: can assign more bits to represent the coarse info, and the remaining bits (if available) to finer details by quantizing the refinement signals accordingly.

Brief Note on Subband vs Wavelet Coding

- The **octave (dyadic)** frequency partition can reflect the **logarithmic** characteristics in human perception.
- Wavelet coding and subband coding have many similarities (e.g. from filter bank perspectives)
  - Traditionally subband coding uses filters that have little overlap to isolate different bands
  - Wavelet transform imposes smoothness conditions on the filters that usually represent a set of basis generated by shifting and scaling (dilation) of a mother wavelet function
  - Wavelet can be motivated from overcoming the poor time-domain localization of short-time FT

$\Rightarrow$ Explore more in Proj#1. See PPV Book Chapter 11

Detailed Derivations

Most General P.R. Conditions (necessary and sufficient)

Recall §1.7.3: sufficient condition for P.R. is $\mathbb{P}(z) = cz^{-m_0}\mathbb{I}$.

① To be free from aliasing, $\mathbb{P}(z)$ must be pseudo circulant.

② $T(z)$ must be a pure delay with possibly a constant multiplicative factor i.e. $T(z) = cz^{-d}$.

$\Longleftrightarrow$ iff the top row of $\mathbb{P}(z)$ is in the form of

$[0, 0 \ldots 0, cz^{-m_0}, 0, \cdots 0]$

Thus $\mathbb{P}(z) = cz^{-m_0} \begin{bmatrix} 0 & I_{M-r} \\ z^{-1} I_r & 0 \end{bmatrix}$ for some $r \in [0, M-1]$

When $r=0$, $\mathbb{P}(z) = \underline{I} \cdot cz^{-m_0}$ as in §1.7.3

The overall transfer function is

$$T(z) = c \cdot z^{-(M-1)} z^{-r} z^{-m_0 M}$$